IMC Follow-up 3

Year 9 Olympiad — Cayley

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

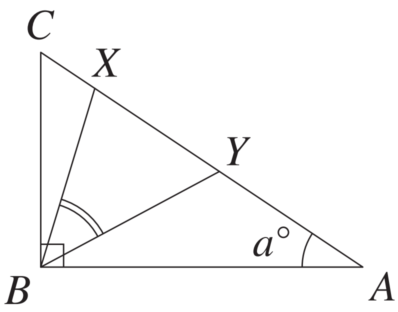
You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Comments and suggestions to DrYuFromShanghai@QQ.com .

1. A triangle, ABC, has a right angle at B and $\angle A = a^\circ$. X lies on AC such that $AX = AB$. Y lies on AC such that $CY = CB$. Prove that $\angle XBY$ has the same size independently of the other angles of the triangle and find the size of that angle.



2. The Intermediate Maths Challenge has 25 questions with the following scoring rules:
5 marks are awarded for each correct answer to Questions 1-15;
6 marks are awarded for each correct answer to Questions 16-25;
Each incorrect answer to Questions 16-20 loses 1 mark;
Each incorrect answer to Questions 21-25 loses 2 marks.
Where no answer is given 0 marks are scored.
Fiona scored 80 marks in total. What possible answers are there to the number of questions Fiona answered correctly?

3. A positive integer $N < 2024$ is divisible by 39 times the sum of its digits. Find all possibilities for N.

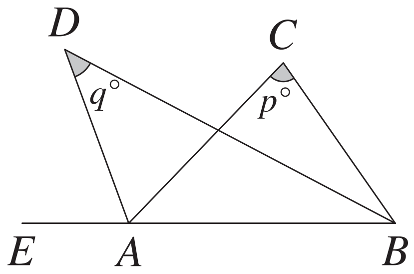
4. When written in ascending order, the nine internal angles from three particular triangles form a sequence where the difference between any adjacent pair of numbers in the sequence is a constant d. One of the angles measures 42°. Find all possible values of the size of the largest of the nine angles.

5. A large number of people arrange themselves into groups of 2, 6 or 10 people. The mean size of a group is 5. However, when each person is asked how many other people are in their group (excluding themselves), the mean of their answers is 7. Prove that there are no groups of 6 people.

6. Into each row of a 9×9 grid, Nigel writes the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 in order, starting at one of the digits and returning to 1 after 9: for example, one row might contain 7, 8, 9, 1, 2, 3, 4, 5, 6. The grid is *gorgeous* if each nine-digit number read along a row or column or along the diagonal from the top-left corner to the bottom-right corner or the diagonal from the bottom-left corner to the top-right corner is divisible by 9. How many of the 9^9 possible grids are *gorgeous*?

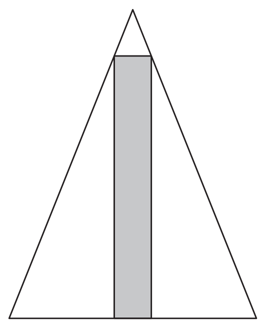
1. A four-digit number, n, is written as '$ABCD$' where A, B, C and D are all different odd digits. It is divisible by each of A, B, C and D. Find all the possible numbers for n.

2. The diagram shows a triangle ABC with side BA extended to a point E. The bisector of $\angle ABC$ meets the bisector of angle $\angle EAC$ at D. Let $\angle BCA = p°$ and $\angle BDA = q°$. Prove that $p = 2q$.



3. Aroon's PIN has four digits. When the first digit (reading from the left) is moved to the end of the PIN, the resulting integer is 6 less than 3 times Aroon's PIN. What could Aroon's PIN be?

4. The diagram shows a rectangle inside an isosceles triangle. The base of the triangle is n times the base of the rectangle, for some integer n greater than 1. Prove that the rectangle occupies a fraction $\frac{2}{n} - \frac{2}{n^2}$ of the total area.



5. The whole numbers from 1 to $2k$ are split into two equal-sized groups in such a way that any two numbers from the same group share no more than two distinct prime factors. What is the largest possible value of k?

6. A bag contains 7 red discs, 8 blue discs and 9 yellow discs. Two discs are drawn at random from the bag. If the discs are the same colour then they are put back into the bag. However, if the discs are different colours then they are removed from the bag and a disc of the third colour is placed in the bag. This procedure is repeated until there is only one disc left in the bag or the only remaining discs in the bag have the same colour. What colour is the last disc (or discs) left in the bag?

1. The numbers 62, 63, 64, 65, 66, 67, 68, 69 and 70 are divided by, in some order, the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9, resulting in nine integers. The sum of these nine integers is S. What are the possible values of S?

2. A palindromic number is a positive integer which reads the same when its digits are reversed, for example 269 962. Find all six-digit palindromic numbers that are divisible by 45.

3. Consider the equation $0.abcd + 0.efgh = 1$, where each letter stands for a digit from 1 to 8 inclusive.

(a) Suppose each letter stands for a different digit. Prove that there are no solutions.

(b) Suppose instead that digits may be repeated. How many solutions are there? (You may give your final answer as a product of prime numbers if you wish.)

Note that $(0.abcd, 0.efgh)$ *and* $(0.efgh, 0.abcd)$ *are considered to be the same solution.*

4. The regular octagon $ABCDEFGH$ is inscribed in a circle. Points P and Q are on the circle, with P between C and D, such that APQ is an equilateral triangle. It is possible to inscribe a regular n-sided polygon, one of whose sides is PD, in the circle. What is the value of n?

5. Consider equations of the form $ax + b = c$, where a, b and c are integers such that one is the sum of the other two and a is non-zero. What are the possible integer values of x?

6. Seth has nine stones: three painted blue, three painted red and three painted yellow. The blue stones are labelled 1, 2 and 3, as are the red stones and the yellow stones. He builds a vertical tower with three stones, putting one on top of another.

Three stones form a *set* if any of the following hold:

(i) They all have the same colour;

(ii) They are all labelled with the same number;

(iii) They all have different colours;

(iv) They are all labelled with different numbers.

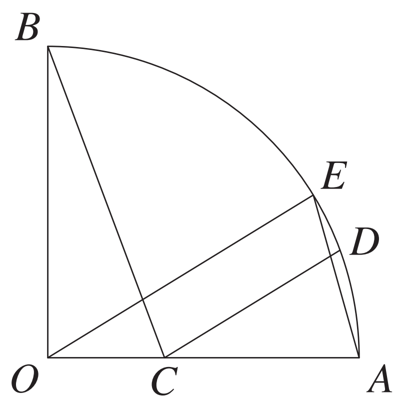
In how many ways can he build a tower that **avoids** creating a set?

1. In the four-digit number 4753, three two-digit numbers are formed by successive pairs of digits $(47, 75, 53)$. Exactly two of these two-digit numbers are prime. Find all four-digit numbers in which all four digits are prime, and **all three** two-digit numbers formed by successive digits are prime.

2. Jack has a large number of tiles, each of which is in the shape of a right-angled triangle with side lengths 3 cm, 4 cm and 5 cm. Is it possible for Jack to combine a number of these tiles, without gaps or overlap, to form a rectangle of size 2016 cm by 2021 cm?

3. In the diagram, OAB is a quarter circle, OE and CD are parallel, BC and EA are parallel, and $\angle BCD = 4 \times \angle OBC$.

What is the size of $\angle OBC$?



4. Let $S(n)$ denote the sum of the first n terms of the series

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots \quad .$$

For example, $S(5) = 1 - 2 + 3 - 4 + 5 = 3$.

For what values of a and b does $S(a) + S(b) + S(a + b) = 1$?

5. Real numbers p, q, r, x, y, z satisfy the equations

$$\frac{x}{p} + \frac{q}{y} = 1,$$

$$\frac{y}{q} + \frac{r}{z} = 1.$$

Prove that $pqr + xyz = 0$.

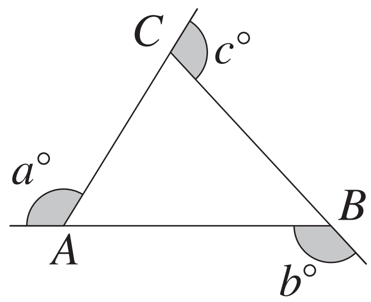
6. During an early morning drive to work, Simon encountered n sets of traffic lights, each set being red, amber or green as he approached it. He noticed that consecutive lights were never the same colour.

Given that he saw *at least* two red lights, find a simplified expression, in terms of n, for the number of possible sequences of colours Simon could have seen.

1. In the triangle ABC, the three exterior angles a°, b° and c° satisfy $a + b = 3c$.

Prove that the triangle ABC is right-angled.

[Note: The diagram has been included to illustrate the labelling only and is not drawn to scale.]



2. The digits 1, 2, 3, 4, 5, A and B are all different and nonzero. Each of the two six-digit integers 'A12345' and '12345A' is divisible by B.

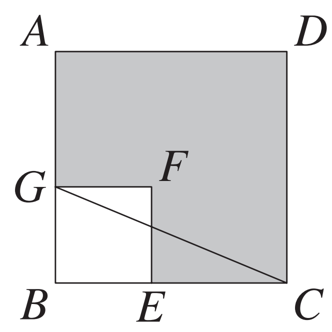
Find all possible pairs of values of A and B.

3. Four friends rent a cottage for a total of £300 for the weekend. The first friend pays half of the sum of the amounts paid by the other three friends. The second friend pays one third of the sum of the amounts paid by the other three friends. The third friend pays one quarter of the sum of the amounts paid by the other three friends.

How much money does the fourth friend pay?

4. Two squares $ABCD$ and $BEFG$ share the vertex B, with E on the side BC and G on the side AB, as shown.
The length of CG is 9 cm and the area of the shaded region is 47 cm^2.

Calculate the perimeter of the shaded region.



5. A ladybird is free to fly between the 1×1 cells of a 10×10 square grid. She may begin in any 1×1 cell of the grid. Every second she flies to a different 1×1 cell that she has not visited before.

Find the smallest number of cells the ladybird must visit, including her starting cell, so that you can be certain that there is a 2×2 grid of adjacent cells, each of which she has visited.

6. Martha and Nadia play a game. Each has to make her own four-digit number, choosing her four digits from eight "digit cards" labelled 1-8. First Martha chooses her thousands digit, and then Nadia chooses her thousands digit. Next, Martha chooses her hundreds digit from the remaining six cards, and then Nadia chooses her hundreds digit. This process is repeated for the tens and finally the units digits of their numbers. The two four-digit numbers produced are then added together. Martha wins if the sum is not a multiple of 6; Nadia wins if the sum is a multiple of 6.

Determine which player has a winning strategy (that is to say, which player can guarantee that she will win no matter which digits the other player chooses).

1. Each of Alice and Beatrice has their birthday on the same day.

In 8 years' time, Alice will be twice as old as Beatrice. Ten years ago, the sum of their ages was 21.

How old is Alice now?

2. In the addition shown, each of the letters D, O, G, C, A and T represents a different digit.

$$\begin{array}{r} D\,O\,G \\ +\,C\,A\,T \\ \hline 1\,0\,0\,0 \end{array}$$

What is the value of $D + O + G + C + A + T$?

3. The triangle ABC is isosceles with $AB = BC$. The point D is a point on BC, between B and C, so that $AC = AD = BD$.

What is the size of angle ABC?

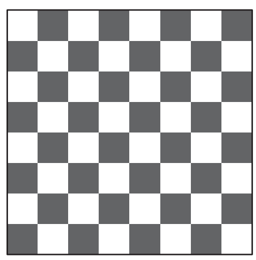
4. Arrange the digits 1, 2, 3, 4, 5, 6, 7, 8 to form two 4-digit integers whose difference is as small as possible.

Explain clearly why your arrangement achieves the smallest possible difference.

5. Howard chooses n different numbers from the list 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, so that no two of his choices add up to a square.

What is the largest possible value of n?

6. A chessboard is formed from an 8×8 grid of alternating black and white squares, as shown. The side of each small square is 1 cm.

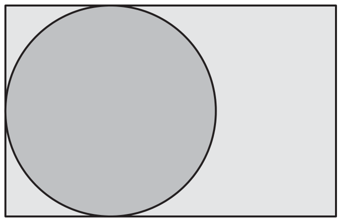


What is the largest possible radius of a circle that can be drawn on the board in such a way that the circumference is entirely on white squares or corners?

C1. The positive integer N has six digits in increasing order. For example, 124 689 is such a number.

However, unlike 124 689, three of the digits of N are 3, 4 and 5, and N is a multiple of 6.
How many possible six-digit integers N are there?

C2. A circle lies within a rectangle and touches three of its edges, as shown.



The area inside the circle equals the area inside the rectangle but outside the circle.

What is the ratio of the length of the rectangle to its width?

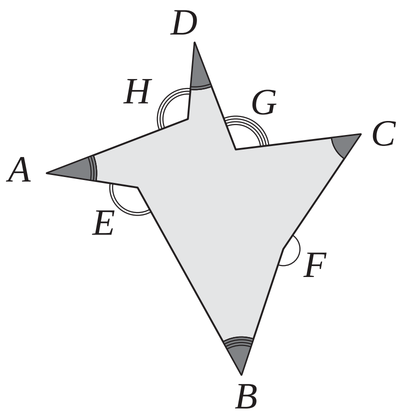
C3. The addition sum $XCV + XXV = CXX$ is true in Roman numerals.

In this question, however, the sum is actually the letter-sum shown alongside, in which: each letter stands for one of the digits 0 to 9, and stands for the same digit each time it occurs; different letters stand for different digits; and no number starts with a zero.

$$\begin{array}{r} XCV \\ +XXV \\ \hline CXX \end{array}$$

Find all solutions, and explain how you can be sure you have found every solution.

C4. Prove that the difference between the sum of the four marked interior angles A, B, C, D and the sum of the four marked exterior angles E, F, G, H of the polygon shown is 360°.

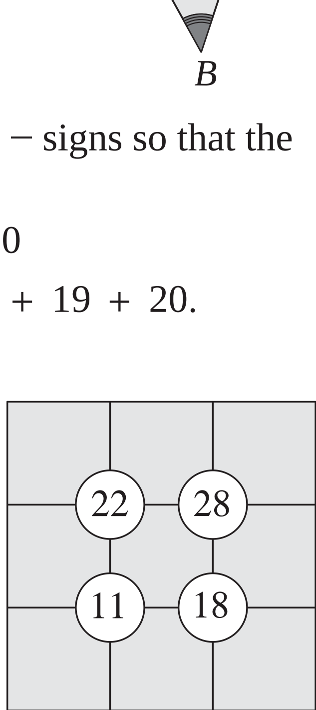


C5. In the expression below, three of the + signs are changed into − signs so that the expression is equal to 100:

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$+ 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

In how many ways can this be done?

C6. In the puzzle *Suko*, the numbers from 1 to 9 are to be placed in the spaces (one number in each) so that the number in each circle is equal to the sum of the numbers in the four surrounding spaces.



How many solutions are there to the *Suko* puzzle shown alongside?