IMC Follow-up 5

Year 11 Olympiad — Maclaurin

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

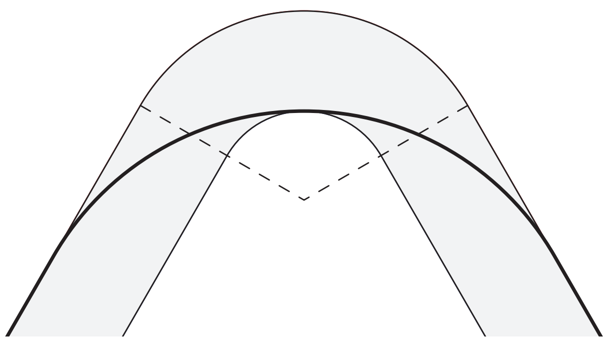
Comments and suggestions to DrYuFromShanghai@QQ.com.

1. Mike is doing a one-hour cycling challenge. He has a computer which predicts how far he will cycle in the rest of the hour based on his average speed so far.
After cycling 1 km in t minutes, he checks the distance the computer predicts he will cycle in the remaining time and it shows d km.
In the next 36 minutes, he cycles 15 km. He checks the computer again and finds it still predicts he will cycle d km in the remaining time.
Find the distance shown on the computer each time he looks.

2. In how many ways can we choose two different integers between -100 and 100 inclusive, so that their sum is greater than their product?

3. What is the smallest number, n, which is the product of 3 distinct primes where the mean of all its factors is not an integer?

4. A bend in a road is formed from two concentric arcs with inside radius r and outside radius R, each of a third of a circle with the same centre. The road is then formed of tangents to the arcs.
A cyclist cuts the corner by following an arc of radius x which is tangent to the outside of the road at its ends and tangent to the inside of the road in the middle.



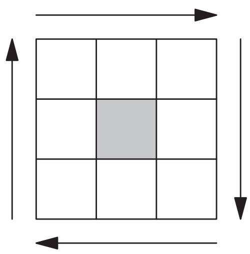
Prove that $r + x = kR$ for some number k to be found.

5. Two right-angled triangles are similar. The larger triangle has short sides which are 1 cm and 5 cm longer than the short sides of the smaller triangle. The area of the larger triangle is $8\,\text{cm}^2$ more than the area of the smaller triangle. Find all possible values for the side lengths of the short sides of the smaller triangle.

6. A busy bee buzzes between the cells of a large honeycomb made up of a plane of tessellated hexagons. A flight of length n consists of picking any of the six neighbouring cells and flying to the n^{th} cell in that direction. After consecutive flights of lengths $n = N, N-1, \ldots, 2, 1$, the bee finds that it has returned to its starting location. For which values of N is this possible?

1. A plank of wood has one end, A, against a vertical wall. Its other end, B, is on horizontal ground. When end A slips down 8cm, end B moves 4cm further away from the wall. When end A slips down a further 9cm, end B moves a further 3cm away from the wall. Find the length of the plank.

2. The digits 1 to 8 are placed into the cells of the grid on the right, making four three-digit numbers when read clockwise. For which values of k from 2 to 6 is it possible to create an arrangement such that all four of the three-digit numbers are multiples of k?



3. $ABCD$ is a square and X is a point on the side DA such that the semicircle with diameter CX touches the side AB. Find the ratio $AX : XD$.

4. The ratio of the number of red beads on a chain to the number of yellow beads is the same as the ratio of the number of yellow beads to the number of blue beads. There are 30 more blue beads than red ones. How many red beads could be on the chain?

5. A 4 by 4 square is divided into sixteen unit cells. Each unit cell is coloured with one of four available colours, red, blue, green or yellow.
The 4 by 4 square contains nine different 2 by 2 “sub-squares". Suppose that we colour the sixteen unit cells in such a way that each 2 by 2 sub-square has one cell of each colour.
Prove that the four corner cells in the large 4 by 4 square must then be coloured differently.

6. Let m, n be fixed positive integers. Prove that there are infinitely many triples of positive integers (x, y, z) such that

$$x^{mn+1} = y^m + z^n$$

for each pair of values (m, n).

1. A fruit has a water content by weight of $m\%$. When left to dry in the sun, it loses $(m-5)\%$ of this water, leaving it with a water content by weight of 50%. What is the value of m?

2. (i) Expand and simplify $(x+1)(x^2-x+1)$.

(ii) Find all powers of 2 that are one more than a cube.

(A power of 2 is a number that can be written in the form 2^n, where n is an integer greater than or equal to 0.)

3. How many distinct triangles satisfy all the following properties:

(i) all three side-lengths are a whole number of centimetres in length;

(ii) at least one side is of length 10 cm;

(iii) at least one side-length is the (arithmetic) mean of the other two side-lengths?

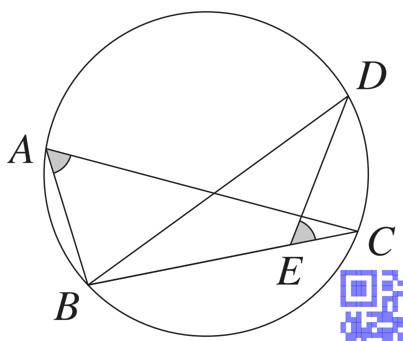
4. A robot sits at the origin of a two-dimensional plane. Each second the robot chooses a direction, North or East, and at the sth second moves 2^{s-1} units in that direction. The total number of moves made by the robot is a multiple of 3. Show that, for each possible total number of moves, there are at least four different routes the robot can take such that the distance from the origin to the robot's final position is an integer.

5. The equation $x^2+bx+c=0$ has two different integer solutions, and the equation $x^2+bx-c=0$ also has two different integer solutions, where b and c are nonzero.

(i) Show that it is possible to find different positive integers p and q such that $2b^2=p^2+q^2$.

(ii) Show that it is possible to find different positive integers r and s such that $b^2=r^2+s^2$.

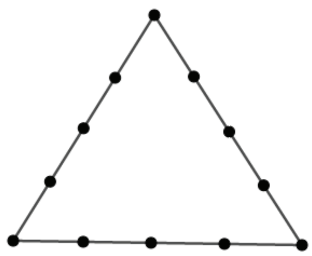
6. The diagram shows a circle with points A, B, C, D on its circumference and point E on chord BC. Given that $\angle BAC = \angle CED$ and $BC = 4 \times CE$, prove that $DB = 2 \times DE$.

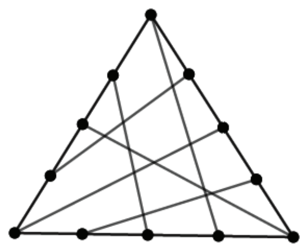


1. Solve the pair of simultaneous equations

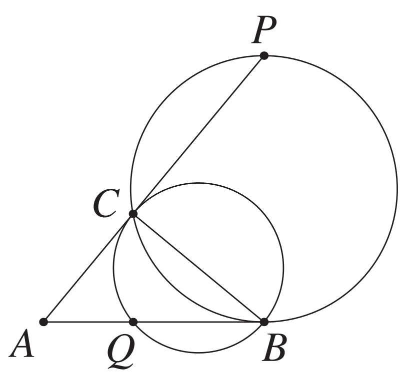
$$x^2 - 2xy = 1,$$
$$5x^2 - 2xy + 2y^2 = 5.$$

2. The 12 points in the first diagram below are to be joined in pairs by 6 line segments that pass through the interior of the triangle. One example is shown in the second diagram. In how many ways can this be done?





3. The diagram shows a triangle ABC. A circle touching AB at B and passing through C cuts the line AC at P. A second circle touching AC at C and passing through B cuts the line AB at Q.



Prove that $\dfrac{AP}{AQ} = \left(\dfrac{AB}{AC}\right)^3$.

4. A sequence of integers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = a_n + 8n \text{ for all integers } n \geq 1.$$

Find all values of k such that every term in the sequence is a square.

5. A triangular playground has sides, in metres, measuring 7, 24 and 25. Inside the playground, a lawn is designed so that the distance from each point on the edge of the lawn to the nearest side is 2 metres. What is the area of the lawn?

6. A cat and a mouse occupy the top right and bottom left cells respectively of an $m \times n$ rectangular grid, where $m, n > 1$. Each second they both move diagonally one cell.

For which pairs (m, n) is it possible for the cat and the mouse to occupy the same cell at the same time?

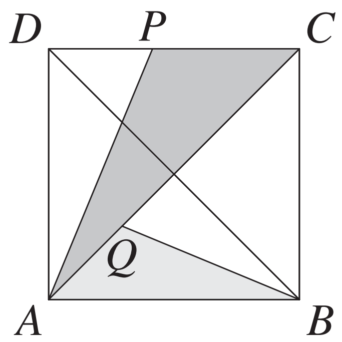
Note: For every pair (m, n) you must either prove that it is impossible for the cat and the mouse to occupy the same cell at the same time, or explain why there is a sequence of moves that ends with the cat and the mouse occupying the same cell at the same time.

1. A bag contains counters, of which ten are coloured blue and Y are coloured yellow. Two yellow counters and some more blue counters are then added to the bag. The proportion of yellow counters in the bag remains unchanged before and after the additional counters are placed into the bag.

Find all possible values of Y.

2. In the square $ABCD$, the bisector of $\angle CAD$ meets CD at P and the bisector of $\angle ABD$ meets AC at Q.

What is the ratio of the area of triangle ACP to the area of triangle BQA?

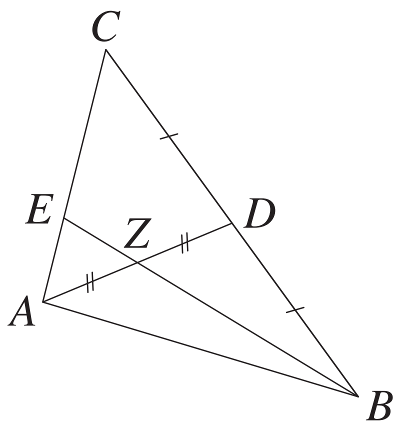


3. An altitude of a triangle is the shortest distance from a vertex to the line containing the opposite side.

Find the side lengths of all possible right-angled triangles with perimeter 5 cm and shortest altitude 1 cm.

4. The diagram shows a triangle ABC and two lines AD and BE, where D is the midpoint of BC and E lies on CA. The lines AD and BE meet at Z, the midpoint of AD.

What is the ratio of the length CE to the length EA?



5. Let p and q respectively be the smallest and largest prime factors of n. Find all positive integers n such that $p^2 + q^2 = n + 9$.

6. Seth and Cain play a game. They take turns, and on each turn a player chooses a pair of integers from 1 to 50. One integer in the pair must be twice the other, and the players cannot choose any integers used previously. The first player who is unable to choose such a pair loses the game. If Seth starts, determine which player, if any, has a winning strategy.

1. A train leaves K for L at 09:30 while another train leaves L for K at 10:00. The first train arrives in L 40 minutes after the trains pass each other. The second train arrives in K 1 hour and 40 minutes after the trains pass.

Each train travels at a constant speed.

At what time did the trains pass each other?

2. A right-angled triangle has area $150\,\text{cm}^2$ and the length of its perimeter is $60\,\text{cm}$.

What are the lengths of its sides?

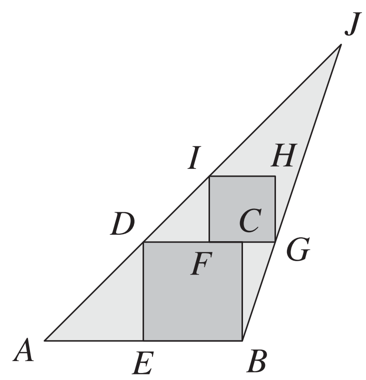
3. Two numbers are such that the sum of their reciprocals is equal to 1. Each of these numbers is then reduced by 1 to give two new numbers.

Prove that these two new numbers are reciprocals of each other.

[*The* reciprocal *of a non-zero number x is the number* $\frac{1}{x}$.]

4. The diagram shows the two squares $BCDE$ and $FGHI$ inside the triangle ABJ, where E is the midpoint of AB and C is the midpoint of FG.

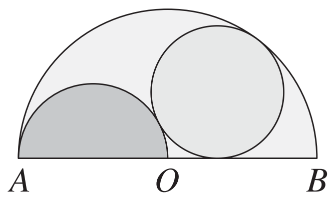
What is the ratio of the area of the square $BCDE$ to the area of the triangle ABJ?



5. A semicircle of radius 1 is drawn inside a semicircle of radius 2, as shown in the diagram, where $OA = OB = 2$.

A circle is drawn so that it touches each of the semicircles and their common diameter, as shown.

What is the radius of the circle?



6. A tiling of an $n \times n$ square grid is formed using 4×1 tiles.

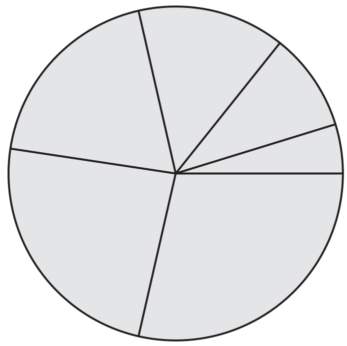
What are the possible values of n?

[*A* tiling *has no gaps or overlaps, and no tile goes outside the region being tiled.*]

M1. The sum of the squares of two real numbers is equal to fifteen times their sum. The difference of the squares of the same two numbers is equal to three times their difference.

Find all possible pairs of numbers that satisfy the above criteria.

M2. The diagram shows a circle that has been divided into six sectors of different sizes.



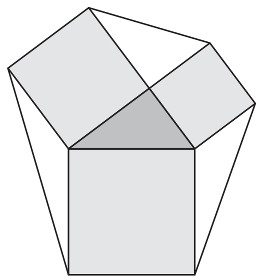
Two of the sectors are to be painted red, two of them are to be painted blue, and two of them are to be painted yellow. Any two sectors which share an edge are to be painted in different colours.

In how many ways can the circle be painted?

M3. Three positive integers have sum 25 and product 360.

Find all possible triples of these integers.

M4. The squares on each side of a right-angled scalene triangle are constructed and three further line segments drawn from the corners of the squares to create a hexagon, as shown. The squares on these three further line segments are then constructed (outside the hexagon).



The combined area of the two equal-sized squares is 2018 cm^2.

What is the total area of the six squares?

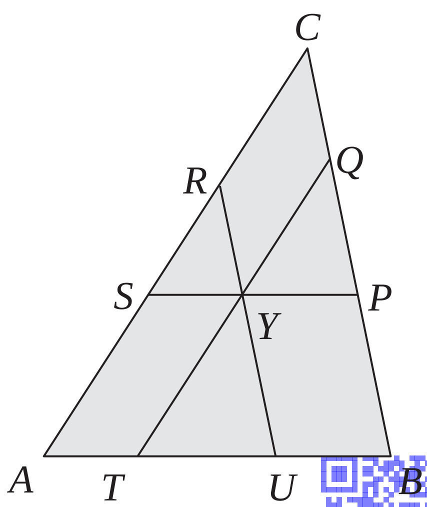
M5. For which integers n is $\dfrac{16\left(n^2 - n - 1\right)^2}{2n - 1}$ also an integer?

M6. The diagram shows a triangle ABC and points T, U on the edge AB, points P, Q on BC, and R, S on CA, where:

(i) SP and AB are parallel, UR and BC are parallel, and QT and CA are parallel;

(ii) SP, UR and QT all pass through a point Y; and

(iii) $PQ = RS = TU$.



Prove that

$$\frac{1}{PQ} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}.$$