IMC Follow-up 4

Year 10 Olympiad — Hamilton

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

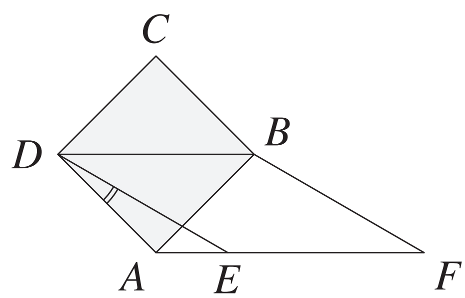
Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Comments and suggestions to DrYuFromShanghai@QQ.com .

1. Richard is cycling at a speed v km/h when he looks at his cycle computer to see how long it will take him to get home at his current speed. It shows a time t hours.
He cycles at this speed for 40 minutes, then instantaneously slows down by 1 km/h and checks his cycle computer; the predicted time to get home at his new speed is still t hours.
After cycling at this new speed for 45 minutes, he instantaneously slows down by another 1 km/h and checks his cycle computer; the predicted time to get home at this speed is again t hours.
How far from home was he when he first looked at the computer?

2. $ABCD$ is a square. $BDEF$ is a rhombus with A, E and F collinear. Find $\angle ADE$.

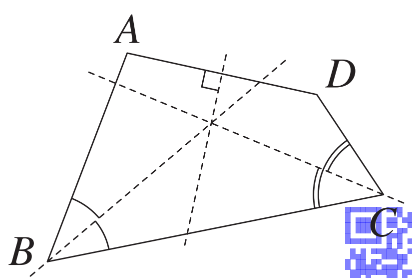


3. A large number of people arrange themselves into groups of 2, 5 or 11 people. The mean size of a group is 4. However, when each person is asked how many other people are in their group (excluding themselves), the mean of their answers is 6. Prove that the number of groups must be a multiple of 27.

4. The numbers 1, 2, 3, 4 and 5 are used once each in some order substituting for the letters in the series of powers $M^{\left(A^{\left(T^{\left(H^{S}\right)}\right)}\right)}$. In how many of the arrangements is the units digit of the value of this expression equal to 1?

5. The integers 1 to 100 are written on a board. Seth chooses two distinct integers from the board, b and c, and forms the quadratic equation $x^2 + bx + c = 0$. If the quadratic equation formed has integer solutions, then he erases b and c from the board; if not, the board remains unchanged.
If Seth continually repeats this process, is it possible for him to erase all the numbers from the board?

6. The diagram shows a quadrilateral $ABCD$ with $AB + CD = BC$. The interior angle bisectors of $\angle B$ and $\angle C$, and the perpendicular bisector of AD, are shown as dotted lines. Prove that those three bisectors meet at a point.

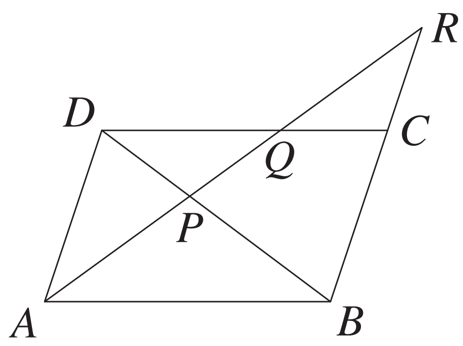


1. Susie thinks of a positive integer n. She notices that, when she divides 2023 by n, she is left with a remainder of 43. Find how many possible values of n there are.

2. The two positive integers a, b with $a > b$ are such that a% of b% of a and b% of a% of b differ by 0.003. Find all possible pairs (a, b).

3. The nth term of a sequence is the first non-zero digit of the decimal expansion of $\frac{1}{\sqrt{n}}$.
How many of the first one million terms of the sequence are equal to 1?

4. In the parallelogram $ABCD$, a line through A meets BD at P, CD at Q and BC extended at R. Prove that $\frac{PQ}{PR} = \left(\frac{PD}{PB}\right)^2$.



5. Mickey writes down on a board n consecutive whole numbers the smallest of which is 2023. He then replaces the largest two numbers on the board with their difference, reducing the number of numbers on the board by one. He does this repeatedly until there is only a single number on the board. For which values of n is this last remaining number 0?

6. Find all triples (m, n, p) which satisfy the equation

$$p^n + 3600 = m^2$$

where p is prime and m, n are positive integers.

1. A regular polygon P has four more sides than another regular polygon Q, and their interior angles differ by $1°$. How many sides does P have?

2. Hudson labels each of the four vertices of a triangular pyramid with a different integer chosen from 1 to 15. For each of the four triangular faces, he then calculates the mean of the three numbers at the vertices of the face.

Given that the means calculated by Hudson are all integers, how many different sets of four numbers could he have chosen to label the vertices of the triangular pyramid?

3. It is possible to write $15129 = 123^2$ as the sum of three distinct squares: $15129 = 27^2 + 72^2 + 96^2$.

(i) By using the identity $(a+b)^2 \equiv a^2+2ab+b^2$, or otherwise, find another way to write 15129 as the sum of three distinct squares.

(ii) Hence, or otherwise, show that $378225 = 615^2$ can be written as the sum of six distinct squares.

4. Mr Evans has a class containing an even number of students. He calculated that in the end-of-term examination the mean mark of the students was 58, the median mark was 80 and the difference between the lowest mark and the highest mark was 40. Show that Mr Evans made a mistake in his calculations.

5. A square $ABCD$ has side-length 2, and M is the midpoint of BC. The circle S inside the quadrilateral $AMCD$ touches the three sides AM, CD and DA. What is its radius?

6. A robot sits at zero on a number line. Each second the robot chooses a direction, left or right, and at the sth second the robot moves 2^{s-1} units in that direction on the number line.

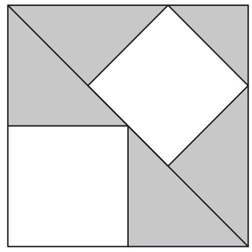
For which integers n are there infinitely many routes the robot can take to reach n?

(You may use the fact that every positive integer can be written as a sum of different powers of 2. For example, $19 = 2^0 + 2^1 + 2^4$*)*

1. Naomi has a broken calculator. All it can do is either add one to the previous answer, or square the previous answer. (It performs the operations correctly.) Naomi starts with 2 on the screen. In how many ways can she obtain an answer of 1000?

2. The diagram shows two unshaded squares inside a larger square.

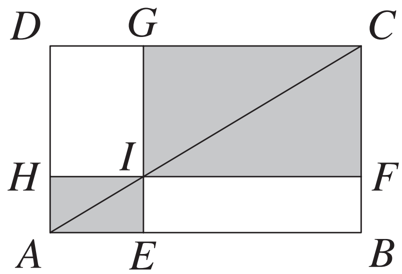
What fraction of the larger square is shaded?



3. For how many positive integers n less than 200 is n^n a cube and $(n+1)^{n+1}$ a square?

4. $ABCD$ is a rectangle with area 6 cm^2.

The point E lies on AB, F lies on BC, G lies on CD and H lies on DA. The point I lies on AC and is the point of intersection of EG and FH, and $AEIH$ and $IFCG$ are both rectangles. One possible diagram is shown to the right.



Given that the combined area of $AEIH$ and $IFCG$ is 4 cm^2, find all possible values for the area of rectangle $AEIH$ in cm^2.

5. Find all real numbers x, y, z such that $x^2 + y^2 + z^2 = x - z = 2$.

6. Humpty buys a box of 15 eggs, with 3 rows and 5 columns. Each meal he removes one egg to cook and eat. If necessary, he moves one or more eggs in the box so that between meals there are always two lines of reflective symmetry. What is the smallest total number of extra egg moves he can make while he empties the box?

Note: You must carefully justify that your answer is minimal; that it is impossible to make fewer extra egg moves while emptying the box.

1. Arun and Disha have some numbered discs to share out between them. They want to end up with one pile each, not necessarily of the same size, where Arun's pile contains exactly one disc numbered with a multiple of 2 and Disha's pile contains exactly one disc numbered with a multiple of 3. For each case below, either count the number of ways of sharing the discs, or explain why it is impossible to share them in this way.

(a) They start with ten discs numbered from 1 to 10.

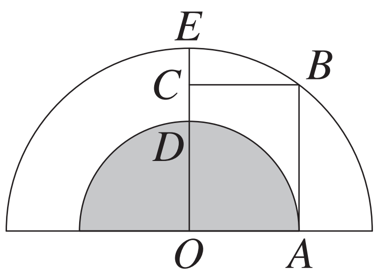
(b) They start with twenty discs numbered from 1 to 20.

2. In the UK, 1p, 2p and 5p coins have thicknesses of 1.6 mm, 2.05 mm and 1.75 mm respectively.

Using only 1p and 5p coins, Joe builds the shortest (non-empty) stack he can whose height in millimetres is equal to its value in pence. Penny does the same but using only 2p and 5p coins.

Whose stack is more valuable?

3. The diagram shows two semicircles with a common centre O and a rectangle $OABC$. The line through O and C meets the small semicircle at D and the large semicircle at E. The lengths CD and CE are equal.

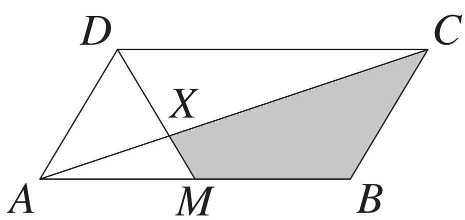


What fraction of the large semicircle is shaded?

4. Piercarlo chooses n integers from 1 to 1000 inclusive. None of his integers is prime, and no two of them share a factor greater than 1.

What is the greatest possible value of n?

5. In the diagram, $ABCD$ is a parallelogram, M is the midpoint of AB and X is the point of intersection of AC and MD.



What is the ratio of the area of $MBCX$ to the area of $ABCD$?

6. We write $\lfloor x \rfloor$ to represent the largest integer less than or equal to x. So, for example, $\lfloor 1.7 \rfloor = 1$, $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$ and $\lfloor -0.4 \rfloor = -1$.

Find all real values of x such that $\lfloor 3x + 4 \rfloor = \lfloor 5x - 1 \rfloor$.

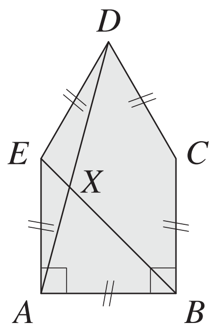
1. A number of couples met and each person shook hands with everyone else present, but not with themselves or their partners.

There were 31 000 handshakes altogether.

How many couples were there?

2. The diagram shows a pentagon $ABCDE$ in which all sides are equal in length and two adjacent interior angles are $90°$. The point X is the point of intersection of AD and BE.

Prove that $DX = BX$.



3. A $4\,\text{cm} \times 4\,\text{cm}$ square is split into four rectangular regions using two line segments parallel to the sides.

How many ways are there to do this so that each region has an area equal to an integer number of square centimetres?

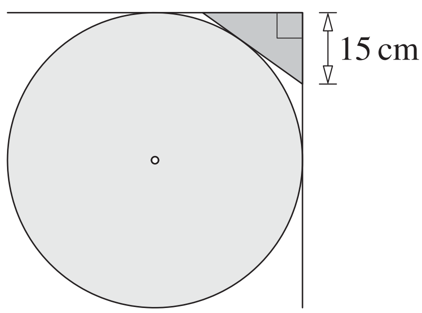
4. Each of A and B is a four-digit palindromic integer, C is a three-digit palindromic integer, and $A - B = C$.

What are the possible values of C?

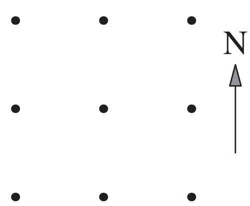
[*A palindromic integer reads the same 'forwards' and 'backwards'.*]

5. The area of the right-angled triangle in the diagram alongside is $60\,\text{cm}^2$. The triangle touches the circle, and one side of the triangle has length 15 cm, as shown.

What is the radius of the circle?



6. Nine dots are arranged in the 2×2 square grid shown. The arrow points north.



Harry and Victoria take it in turns to draw a unit line segment to join two dots in the grid.

Harry is only allowed to draw an east-west line segment, and Victoria is only allowed to draw a north-south line segment. Harry goes first.

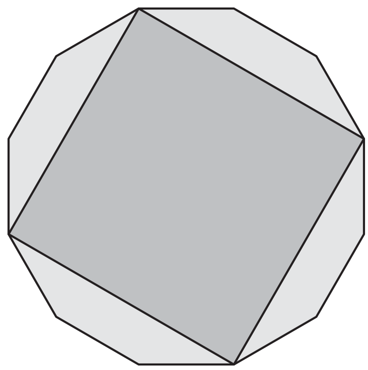
A point is scored when a player draws a line segment that completes a 1×1 square on the grid.

Can either player force a win, no matter how the other person plays?

H1. The positive integers m and n satisfy the equation $20m + 18n = 2018$. How many possible values of m are there?

H2. How many nine-digit integers of the form '$pqrpqrpqr$' are multiples of 24? (*Note that p, q and r need not be different.*)

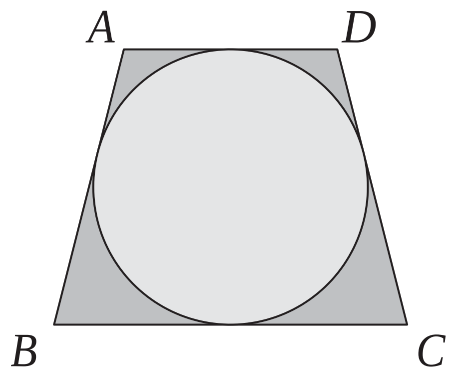
H3. The diagram shows a regular dodecagon and a square, whose vertices are also vertices of the dodecagon.



What is the value of the ratio

area of the square : area of the dodecagon?

H4. The diagram shows a circle and a trapezium $ABCD$ in which AD is parallel to BC and $AB = DC$. All four sides of $ABCD$ are tangents of the circle. The circle has radius 4 and the area of $ABCD$ is 72.

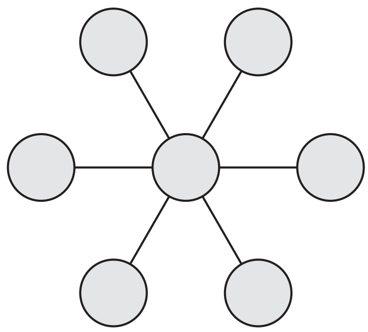


What is the length of AB?

H5. A two-digit number is divided by the sum of its digits. The result is a number between 2.6 and 2.7.

Find all of the possible values of the original two-digit number.

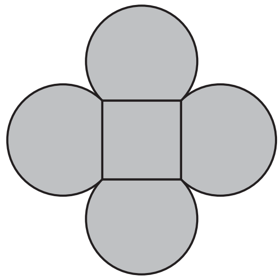
H6. The figure shows seven circles joined by three straight lines.



The numbers 9, 12, 18, 24, 36, 48 and 96 are to be placed into the circles, one in each, so that the product of the three numbers on each of the three lines is the same.

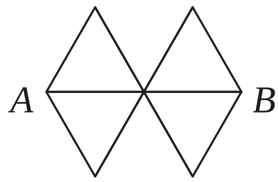
Which of the numbers could go in the centre?

H1. The diagram shows four equal arcs placed on the sides of a square. Each arc is a major arc of a circle with radius 1 cm, and each side of the square has length $\sqrt{2}$ cm.



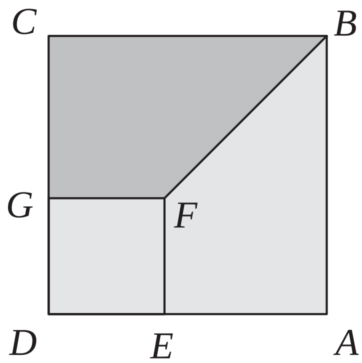
What is the area of the shaded region?

H2. A ladybird walks from A to B along the edges of the network shown. She never walks along the same edge twice. However, she may pass through the same point more than once, though she stops the first time she reaches B.



How many different routes can she take?

H3. The diagram shows squares $ABCD$ and $EFGD$. The length of BF is 10 cm. The area of trapezium $BCGF$ is 35 cm^2.



What is the length of AB?

H4. The largest of four different real numbers is d. When the numbers are summed in pairs, the four largest sums are 9, 10, 12 and 13.

What are the possible values of d?

H5. In the trapezium $ABCD$, the lines AB and DC are parallel, $BC = AD$, $DC = 2AD$ and $AB = 3AD$.
The angle bisectors of $\angle DAB$ and $\angle CBA$ intersect at the point E.

What fraction of the area of the trapezium $ABCD$ is the area of the triangle ABE?

H6. Solve the pair of simultaneous equations

$$x^2 + 3y = 10 \qquad \text{and}$$

$$3 + y = \frac{10}{x}.$$